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# $L_1$ -Estimates for Eigenfunctions of the Dirichlet Laplacian

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## Abstract

For  $d \in \mathbb{N}$  and  $\Omega \neq \emptyset$  an open set in  $\mathbb{R}^d$ , we consider the eigenfunctions  $\Phi$  of the Dirichlet Laplacian  $-\Delta_\Omega$  of  $\Omega$ . If  $\Phi$  is associated with an eigenvalue below the essential spectrum of  $-\Delta_\Omega$  we provide estimates for the  $L_1$ -norm of  $\Phi$  in terms of its  $L_2$ -norm and spectral data. These  $L_1$ -estimates are then used in the comparison of the heat content of  $\Omega$  at time  $t > 0$  and the heat trace at times  $t' > 0$ , where a two-sided estimate is established. We furthermore show that all eigenfunctions of  $-\Delta_\Omega$  which are associated with a discrete eigenvalue of  $H_\Omega$ , belong to  $L_1(\Omega)$ .

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## Introduction

We study the eigenfunctions  $\Phi$  of the Dirichlet Laplacian  $H_\Omega$  on an open set  $\Omega \subseteq \mathbb{R}^d$ , associated with a (discrete) eigenvalue  $\lambda \in \mathbb{R}$ . Our main interest is to provide bounds on  $\|\Phi\|_1$ , the norm of  $\Phi$  in  $L_1(\Omega)$ , in terms of the  $L_2$ -norm of  $\Phi$  and spectral data. In many cases this is an improvement over the elementary estimate  $\|u\|_1^2 \leq \text{vol}(\Omega)\|u\|_2^2$ , valid for  $\Omega$  of finite volume (and all functions  $u \in L_2(\Omega)$ , not just eigenfunctions). Roughly speaking, we advocate here to replace the factor  $\text{vol}(\Omega)$  with  $\lambda_1^{-d/2} N_{2\lambda_k}(H_\Omega)$ , where  $\lambda_1$  denotes the lowest eigenvalue of  $H_\Omega$  and  $N_t$  counts the (repeated) eigenvalues of  $H_\Omega$  less than or equal to  $t$ . Our actual estimates are more complicated than that, and they only hold for eigenvalues below the essential spectrum. That spectral information can be used instead of volume doesn't come as a complete surprise: indeed, the uncertainty principle

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has found various expressions in spectral terms as in Weyl's Law and other well-known results that connect volumes in phase space with the counting of eigenvalues [13]. Estimates for the  $L_1$ -norm of eigenfunctions as presented here have been a desideratum for several decades now because they yield bounds on the heat content of  $\Omega$  in terms of the heat trace; see also [4; p. 2065].

Throughout this paper,  $H_\Omega$  will be defined as the Friedrichs extension of  $-\Delta$  on  $C_c^\infty(\Omega)$ ;  $H_\Omega$  is self-adjoint and non-negative. More precisely, the form domain of  $H_\Omega$  is given by the Sobolev space  $H_0^1(\Omega)$ , and  $H_\Omega$  satisfies

$$\langle H_\Omega u, v \rangle = \int_\Omega \nabla u \cdot \nabla v \, dx, \quad (0.1)$$

for all  $u \in D(H_\Omega)$  and all  $v \in H_0^1(\Omega)$ . The eigenfunctions  $\Phi$  of  $H_\Omega$ , associated with an eigenvalue  $\lambda$ , are smooth and bounded and obey a well-known estimate

$$\|\Phi\|_\infty^2 \leq C \lambda^{d/2} \|\Phi\|_2^2, \quad (0.2)$$

with a constant  $C$  depending on  $d$  only. Theorem 1.6 below gives an explicit constant. This estimate is a direct consequence of the domain monotonicity of the heat kernel [8]. Interpolation then yields bounds on  $\|\Phi\|_q$  for any  $q \in (2, \infty)$ . An immediate consequence of (0.2) is a lower bound for  $\|\Phi\|_1$  of the form

$$\|\Phi\|_1^2 \geq C \lambda^{-d/2} \|\Phi\|_2^2, \quad (0.3)$$

where  $C$  is a strictly positive constant depending on  $d$  only. Note that the estimates (0.2) and (0.3) hold for *all* eigenvalues.

We now complement the  $L_\infty$ -estimate (0.2) by upper bounds on  $\|\Phi\|_1$ , where we have to make the stronger assumption that  $\Phi$  is an eigenfunction associated with a discrete eigenvalue  $\lambda_k$  located below the essential spectrum of  $H_\Omega$ . Here the  $\lambda_k$  are numbered in increasing order and repeated according to their respective multiplicity. One of our basic estimates reads as follows:

**0.1 Theorem.** *For any  $d \in \mathbb{N}$  there exists a constant  $C$  (depending on  $d$  only) with the following property: If  $\Omega \neq \emptyset$  is an open subset of  $\mathbb{R}^d$  with  $\sigma_{\text{ess}}(H_\Omega) = \emptyset$ , we have*

$$\|\Phi\|_1^2 \leq C \lambda_1^{-d/2} \left( \left( \frac{\lambda_k}{\lambda_1} \right)^d (\log N_{2\lambda_k}(H_\Omega))^d N_{2\lambda_k}(H_\Omega) + \left( \frac{\lambda_k}{\lambda_1} \right)^{4d-3} \right) \|\Phi\|_2^2, \quad (0.4)$$

for all eigenfunctions  $\Phi$  of  $H_\Omega$  associated with the eigenvalue  $\lambda_k$ .

A slightly more general version is given in Corollary 1.4. The estimate (0.4) is structured in the following way: there is a global factor  $\lambda_1^{-d/2}$  up front which comes from scaling; this factor appears if one looks at the ground state eigenfunctions of balls of radius  $r > 0$ . The factors of  $\lambda_k/\lambda_1$  deal with the behavior of the estimate for large eigenvalues as compared to small eigenvalues. The discrepancy between the lower bound (0.3) and the upper bound (0.4) is, at least in part, due to the fact that the lower bound (0.3) seems to be far off in situations with large clouds of

eigenvalues close to  $\lambda_k$ , as can be seen in simple examples like our Example 1.8(3). This difficulty can be overcome by introducing the eigenvalue counting function  $N_\lambda(H_\Omega)$  into the estimates. An improved estimate—in particular avoiding the second term on the right hand side in (0.4)—has recently been obtained by H. Vogt [26] by methods which are partially different from ours.

There are similar, but somewhat more complicated estimates for the case where  $\sigma_{\text{ess}}(H_\Omega) \neq \emptyset$  and  $\lambda_k < \inf \sigma_{\text{ess}}(H_\Omega)$ ; cf. Corollary 1.7. Along these lines it would also be possible to give estimates for  $\|\Phi\|_1$  that involve the gap length  $\lambda_{k+1} - \lambda_k$ , provided  $\lambda_{k+1} > \lambda_k$ , for eigenfunctions  $\Phi$  of  $H_\Omega$  associated with the eigenvalue  $\lambda_k$ . We refer to the comment after Corollary 1.4. We emphasize that our bounds do *not* depend on the volume of  $\Omega$ .

We furthermore show that the eigenfunctions  $\Phi$  of  $H_\Omega$  belong to  $L_1(\Omega)$  if they are associated with a discrete eigenvalue  $\lambda$ , and so  $\Phi \in \bigcap_{1 \leq p < \infty} L_p(\Omega)$ . There is no reason to expect a similar result for eigenfunctions associated with an eigenvalue which belongs to the essential spectrum; see e.g. Example 1.8(2) below.

Let us briefly indicate why estimates as in Theorem 0.1 are possible. They are essentially based on three facts which we describe in terms of a covering of  $\mathbb{R}^d$  by cubes  $Q_{n,j}$  (for  $n \in \mathbb{N}$  and  $j \in \mathbb{Z}^d$ ), where each  $Q_{n,j}$  has edge length  $2n$  and is centered at  $nj$ . Let us consider  $\Omega$ ,  $\lambda_k$  and  $\Phi$  as in Theorem 0.1.

(1) We first observe that there can only be a finite number of cubes  $Q_{n,j}$  such that the Dirichlet Laplacian of  $\Omega \cap Q_{n,j}$  has eigenvalues below  $2\lambda_k$ . In fact, the number of these cubes can be estimated in terms of  $N_{2\lambda_k}$ . We let  $F_n$  denote the union of these cubes. The contribution to the  $L_1$ -norm of  $\Phi$  coming from  $F_n$  (or a slightly larger set) can now be estimated in terms of  $\text{vol}(F_n)$  and, thus, in terms of  $n$  and  $N_{2\lambda_k}$ .

(2) Letting  $G_n := \Omega \setminus F_n$ , we use a partition of unity (subordinate to the covering by the cubes  $Q_{n,j}$ ) and the IMS-localization formula to show that the Dirichlet Laplacian of  $G_n$  has no eigenvalues below  $3\lambda_k/2$ .

(3) The third fact concerns the decay of the eigenfunctions  $\Phi$  associated with the eigenvalue  $\lambda_k$  as we move away from  $F_n$ . Here we use a rather precise, quantitative version of exponential decay which takes into account the distance from the set  $F_n$ . A standard decay estimate holding just outside some large ball containing  $F_n$  would clearly be insufficient for our purposes. Since exponential decay takes place on a larger scale we also have to introduce the sets  $\tilde{F}_n$  and  $\tilde{\tilde{F}}_n$  that are somewhat larger than  $F_n$ .

The estimate given in Theorem 0.1 can be applied in the comparison of the *heat content*  $Q_\Omega$  of an open set  $\Omega \subseteq \mathbb{R}^d$ ,

$$Q_\Omega(t) := \int_\Omega \int_\Omega p_\Omega(x, y; t) \, dy \, dx$$

at time  $t > 0$ , where  $p_\Omega: \Omega \times \Omega \times (0, \infty) \rightarrow [0, \infty)$  is the Dirichlet heat kernel for  $\Omega$ , and the *heat trace*  $Z_\Omega$ ,

$$Z_\Omega(t) := \sum_{k=1}^{\infty} e^{-\lambda_k t}$$

at time  $t > 0$ , where it is assumed that  $H_\Omega$  has compact resolvent and that  $(\lambda_k)$  is

the sequence of all eigenvalues of  $H_\Omega$ . We shall show in Section 5 that  $Q_\Omega(t) < \infty$  for all  $t > 0$  is equivalent to  $Z_\Omega(t) < \infty$  for all  $t > 0$ , and that there is a two-sided estimate. Note that our upper bound for  $Q_\Omega(t)$  involves  $Z_\Omega(t/2)$  and  $Z_\Omega(t/6)^3$ . The interest in a bound on  $Q_\Omega$  in terms of  $Z_\Omega$  lies in the fact that the quantity  $Z_\Omega$  is much simpler (and also simpler to compute) than  $Q_\Omega$  because only information from the Hilbert space  $L_2(\Omega)$  is needed.

Similar estimates on the  $L_1$ -norm of eigenfunctions of Schrödinger operators will be the subject of a forthcoming paper. It would also be of great interest and importance to generalize our results to the case of domains on Riemannian manifolds with sub-exponential growth at infinity [25]. The case of hyperbolic manifolds poses different challenges as can be seen from [9].

Our paper is organized as follows. In Section 1 we first consider the case where  $\lambda_1(H_\Omega) = 1$  and present a basic estimate in its most general form (viz. Proposition 1.1); the proof of Proposition 1.1 is deferred to Section 2. We then derive several estimates from Proposition 1.1 by scaling and a judicious choice of parameters in Proposition 1.1. In Section 2 we construct IMS-partitions of unity, depending on a parameter  $n$ , and prove Proposition 1.1; here we rely on an exponential decay estimate stated in Lemma 2.3. Section 3 is devoted to a proof of Lemma 2.3.

In Section 4 we combine some results from the theory of the Laplacian in  $L_p(\Omega)$ , defined as the generator of the heat semigroup acting in  $L_p(\Omega)$ , to show that the Riesz projection, associated with the eigenspace of a discrete eigenvalue, is independent of  $p$ , for  $1 \leq p < \infty$ . It is then easy to conclude that the range of this projection must be contained in  $L_1(\Omega)$ . This part of the paper has been motivated by the work [16, 17] of two of the authors on the  $L_p$ -spectrum of Schrödinger operators.

In Section 5, finally, we discuss a two-sided estimate for the heat content and the heat trace, using the kernel  $p_\Omega(x, y; t)$  of the heat semigroup. We also give a proof of the lower bound of Theorem 1.6.

**Disclaimer.** In much of this text we let  $C$  denote a generic non-negative constant the value of which may change from line to line.

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## 1 Estimates for the $L_1$ -norm of eigenfunctions

Let  $d \in \mathbb{N}$ . For an open set  $\emptyset \neq \Omega \subseteq \mathbb{R}^d$  we let  $H_\Omega$  denote the (self-adjoint and non-negative) Dirichlet Laplacian of  $\Omega$ , i.e.,  $H_\Omega$  is the unique self-adjoint operator with form domain given by the Sobolev space  $H_0^1(\Omega)$ , and satisfying

$$\langle H_\Omega u, v \rangle = \langle \nabla u, \nabla v \rangle \quad (u \in D(H_\Omega), v \in H_0^1(\Omega)),$$

where  $D(H_\Omega) \subseteq H_0^1(\Omega)$  denotes the domain of  $H_\Omega$ . By construction,  $C_c^\infty(\Omega)$  is a form core of  $H_\Omega$ . Furthermore,  $D(H_\Omega) \subseteq H_{\text{loc}}^2(\Omega)$  and  $H_\Omega u = -\Delta u \in L_2(\Omega)$  for any  $u \in D(H_\Omega)$ . In general,  $D(H_\Omega)$  need not be contained in the Sobolev space  $H^2(\Omega)$ .

**Definition.** For  $\Lambda > 0$ , we let  $\mathcal{O}_\Lambda$  denote the set of all open sets  $\Omega \subseteq \mathbb{R}^d$  that enjoy the property

$$\Lambda = \inf \sigma(H_\Omega) < \inf \sigma_{\text{ess}}(H_\Omega).$$

The sets  $\Omega \in \mathcal{O}_\Lambda$  may be unbounded, and they may have infinite volume. Also, they may consist of countably many components. No regularity of the boundary  $\partial\Omega$  will be required.

The spectrum of  $H_\Omega$  below  $\Sigma_\Omega := \inf \sigma_{\text{ess}}(H_\Omega)$  is purely discrete. It consists of a countable set of eigenvalues  $\lambda_k(H_\Omega)$  (with  $1 \leq k \leq K$  for some  $K \in \mathbb{N}$ , or for  $k \in \mathbb{N}$ ), which we assume to be numbered such that

$$\Lambda = \lambda_1(H_\Omega) \leq \lambda_2(H_\Omega) \leq \dots \leq \lambda_k(H_\Omega) \leq \lambda_{k+1}(H_\Omega) < \Sigma_\Omega,$$

where each eigenvalue is repeated according to its multiplicity. If there is an infinite number of eigenvalues, we have  $\lambda_k \rightarrow \Sigma_\Omega$  if  $\sigma_{\text{ess}}(H_\Omega) \neq \emptyset$ , and  $\lambda_k \rightarrow \infty$  if  $H_\Omega$  has compact resolvent. If  $\Omega$  is connected the ground state eigenfunction is unique (up to scalar multiples) and  $\lambda_1 < \lambda_2$ .

Our results pertain in particular to the case where  $H_\Omega$  has compact resolvent. If  $\Omega$  has finite volume then  $H_\Omega$  has compact resolvent. Furthermore it is well-known that even if  $\Omega$  has infinite volume  $H_\Omega$  may have compact resolvent. Necessary and sufficient criteria for  $H_\Omega$  to have compact resolvent in terms of  $\Omega$  have been obtained in Section 15.7.3 of [20] and in [21]. For concrete estimates for the counting function we refer to [3] and the references therein. We also refer to the elementary Example 1.8(1) below.

Note that, for  $\Lambda > 0$ , the set  $\mathcal{O}_\Lambda$  can be obtained from the set  $\mathcal{O}_1$  by scaling;

$$\Omega \in \mathcal{O}_\Lambda \iff \sqrt{\Lambda} \Omega \in \mathcal{O}_1,$$

for all  $\Lambda > 0$ . Therefore, we will first derive an estimate on  $\|\Phi\|_1$  for  $\Omega$  in the set  $\mathcal{O}_1$ . The general result will then easily follow by scaling. In our basic estimate for the set  $\mathcal{O}_1$  we will work with parameters  $r, t$  satisfying

$$1 \leq r < t < \Sigma_\Omega;$$

as usual, we let  $\Sigma_\Omega = \inf \sigma_{\text{ess}}(H_\Omega) = \infty$  if  $\sigma_{\text{ess}}(H_\Omega) = \emptyset$ .

Below, we will derive estimates for all eigenfunctions  $\Phi$  of  $H_\Omega$  associated with eigenvalues  $\lambda_k \in [1, r]$ . These estimates will depend on the number of eigenvalues of  $H_\Omega$  in the interval  $[1, t]$ , counting multiplicities. Here we use the following definition: for a self-adjoint operator  $T$  and  $t \in \mathbb{R}$  we write

$$N_t(T) := \text{tr } E_T((-\infty, t]),$$

where  $\text{tr}$  denotes the trace, and  $E_T(I)$  is the spectral projection of  $T$  associated with the interval  $I \subseteq \mathbb{R}$ . In particular, if  $T$  is semi-bounded from below and if  $t < \inf \sigma_{\text{ess}}(T)$ , then  $N_t(T)$  denotes the number of eigenvalues of  $T$  less than or

equal to  $t$ , counting multiplicities. If, in our enumeration of eigenvalues,  $\lambda_{k+1} > \lambda_k$  for some  $k$ , we have  $N_{\lambda_k}(T) = k$ .

In order to express a certain quantity occurring in the estimate derived below, we fix (throughout the whole paper) a function  $\varrho \in C_c^\infty(\mathbb{R}^d)$ ,  $\varrho \geq 0$ , with  $\text{spt } \varrho \subseteq B(0, 1/2)$  and  $\int \varrho(x) dx = 1$ , and we define

$$m_0 := \max\{1, \|\Delta \varrho\|_1\}. \quad (1.1)$$

The following proposition contains our basic estimate for sets  $\Omega \in \mathcal{O}_1$ . In the statement we will use, for given  $1 \leq r < t$ , the quantities  $\alpha$  and  $\beta$  (depending on  $r$  and  $t$ ) defined by

$$\beta := (t - r)/2$$

and

$$\alpha := \frac{\min\{\beta, 1\}}{16m_0r}, \quad (1.2)$$

with  $m_0$  from (1.1).

**1.1 Proposition.** *For any  $d \in \mathbb{N}$  there exist constants  $C, c > 0$ , such that for all  $\Omega \in \mathcal{O}_1$ , for all  $1 \leq r < t < \inf \sigma_{\text{ess}}(H_\Omega)$ , and for all  $n \geq \max\{1, 2^{d/2}c/\sqrt{\beta}\}$  we have*

$$\|\Phi\|_1 \leq C \left( n^{d/2} \sqrt{N_t(H_\Omega)} + \frac{\sqrt{r}}{\beta} \left( \frac{n^{2d-2}}{\alpha} + \frac{n^{d-1}}{\alpha^d} \right) e^{-\alpha n} N_t(H_\Omega) \right) \|\Phi\|_2, \quad (1.3)$$

for all eigenfunctions  $\Phi$  of  $H_\Omega$  associated with an eigenvalue  $\lambda_k(H_\Omega) \in [1, r]$ .

The presence of  $N_t = N_t(H_\Omega)$  takes care of situations where there is a “cloud” of eigenvalues below  $t$ . Examples of dumb-bell type show that at least a factor  $\sqrt{N_t}$  appears to be necessary. We emphasize that the trivial estimate, valid for all sets  $\Omega$  of bounded volume,  $\|\Phi\|_1 \leq |\Omega|^{1/2} \|\Phi\|_2$ , is often inadequate, and an estimate in terms of spectral data seems to be more appropriate and desirable.

The constant  $c$  appearing in the assumptions of Proposition 1.1 depends solely on the IMS-partition of unity  $(\Psi_j)_{j \in \mathbb{Z}^d}$  introduced at the beginning of Section 2. The partition of unity can be constructed in such a way that the constant  $c$  is easy to compute. The constant  $C$  appearing on the right hand side of (1.3) could be explicitly computed as a function of the dimension  $d$ . A proof of Proposition 1.1 will be given in Section 2.

In the next step we will reduce the number of free parameters specifying  $n$  first.

**1.2 Theorem.** *(Case of  $\mathcal{O}_1$ )*

*For any  $d \in \mathbb{N}$  there exists a constant  $C$  (depending on  $d$  only) such that for any  $\Omega \in \mathcal{O}_1$  and any  $r, t \in [1, \Sigma_\Omega)$  with  $r < t \leq 3r$ , we have*

$$\|\Phi\|_1^2 \leq C \left( \left( \frac{r^2}{t-r} \right)^d (\log N_t)^d N_t + r^{-3} \left( \frac{r^2}{t-r} \right)^{4d} \right) \|\Phi\|_2^2 \quad (1.4)$$

for all eigenfunctions  $\Phi$  of  $H_\Omega$  associated with an eigenvalue  $\lambda_k \in [1, r]$ .

*Proof.* With  $\eta := \frac{t-r}{2r}$  we obtain  $0 < \eta \leq 1$ ,  $t = (1 + 2\eta)r$  and  $\beta = \eta r$ . We will apply Proposition 1.1 with two different choices of  $n$ . We will also use the elementary estimate  $1 \leq 1/\alpha \leq 16m_0 \frac{r}{\eta}$ .

(1) For  $\log N_t \leq \max\{1, 2^{d/2-1}c\}$ , the choice  $n := \max\{1, 2^{d/2}c/\sqrt{\beta}\}$  yields

$$\begin{aligned} \|\Phi\|_1 &\leq C \max\left\{r^{d-1/2}\eta^{-d-1}, r^{d/2}\eta^{-(3d+1)/2}\right\} \|\Phi\|_2 \\ &\leq Cr^{d-1/2}\eta^{-(3d+1)/2} \|\Phi\|_2. \end{aligned} \quad (1.5)$$

(As a hint for the computation we mention that it is advantageous to write

$$\frac{\sqrt{r}}{\beta} \left( \frac{n^{2d-2}}{\alpha} + \frac{n^{d-1}}{\alpha^d} \right) = \frac{\sqrt{r}}{\alpha\beta} (n^{2d-2} + (n/\alpha)^{d-1})$$

and to note that  $1/(\alpha\beta) \leq C/\eta^2$ .)

(2) For  $\log N_t \geq \max\{1, 2^{d/2-1}c\}$  we choose  $n := 2^{\frac{\log N_t}{\alpha}}$ , where we note that the condition  $n \geq \max\{1, 2^{d/2}c/\sqrt{\beta}\}$  is easily verified. We then have  $e^{-\alpha n} N_t = 1/N_t$  so that

$$(\log N_t)^k \cdot e^{-\alpha n} \cdot N_t \leq C_k \quad (k \in \mathbb{N}),$$

where  $C_k := \max\{(\log u)^k u^{-1}; u \geq 1\}$ .

In the first term on the right hand side of (1.3) we simply estimate  $n^{d/2} \leq C(r/\eta)^{d/2}(\log N_t)^{d/2}$ . As for the second term in the right hand side of (1.3) we first observe that

$$\frac{n^{2d-2}}{\alpha} + \frac{n^{d-1}}{\alpha^d} = \frac{1}{\alpha^{2d-1}} (2^{2d-2}(\log N_t)^{2d-2} + 2^{d-1}(\log N_t)^{d-1})$$

so that

$$\frac{\sqrt{r}}{\beta} \left( \frac{n^{2d-2}}{\alpha} + \frac{n^{d-1}}{\alpha^d} \right) e^{-\alpha n} N_t \leq \frac{C}{\sqrt{r}\eta} (r/\eta)^{2d-1}$$

with  $C := (16m_0)^{2d-1} (2^{2d-2}C_{2d-2} + 2^{d-1}C_{d-1})$ . Now Proposition 1.1 gives

$$\|\Phi\|_1 \leq C \left( (r/\eta)^{d/2} (\log N_t)^{d/2} \sqrt{N_t} + r^{-3/2} (r/\eta)^{2d} \right) \|\Phi\|_2. \quad (1.6)$$

(3) Note that the estimate (1.5) implies (1.6). Inserting the definition of  $\eta$  and taking squares one obtains (1.4).  $\blacksquare$

We now pass from the set  $\mathcal{O}_1$  to the sets  $\mathcal{O}_\Lambda$ , with  $\Lambda > 0$ , by a straightforward scaling argument.

**1.3 Theorem.** (Case of  $\mathcal{O}_\Lambda$ ) For any  $d \in \mathbb{N}$  there exists a constant  $C$  (depending on  $d$  only) such that for any  $\Lambda > 0$  and  $\Omega \in \mathcal{O}_\Lambda$  the following estimate holds: If  $r, t \in [\Lambda, \Sigma_\Omega]$  satisfy  $r < t \leq 3r$ , we have

$$\|\Phi\|_1^2 \leq C \Lambda^{-d/2} \left( \left( \frac{r^2}{\Lambda(t-r)} \right)^d (\log N_t)^d N_t + \left( \frac{r}{\Lambda} \right)^{-3} \left( \frac{r^2}{\Lambda(t-r)} \right)^{4d} \right) \|\Phi\|_2^2,$$

for all eigenfunctions  $\Phi$  of  $H_\Omega$  associated with an eigenvalue  $\lambda_k \in [\Lambda, r]$ .



As will become clear later on, the factor  $\frac{r^2}{\Lambda(t-r)}$  in the above theorem should be read as  $\frac{r}{\Lambda} \cdot \frac{r}{t-r}$ .

*Proof.* Let  $\tilde{\Omega} := \sqrt{\Lambda}\Omega$  ( $\in \mathcal{O}_1$ ). Then  $\inf \sigma_{\text{ess}}(H_{\tilde{\Omega}}) = \frac{1}{\Lambda}\Sigma_{\Omega}$  and to each eigenvalue  $\lambda_k$  of  $H_{\Omega}$  below  $\Sigma_{\Omega}$  there corresponds precisely one eigenvalue  $\tilde{\lambda}_k$  of  $H_{\tilde{\Omega}}$  below  $\frac{1}{\Lambda}\Sigma_{\Omega}$ ; in fact,

$$\tilde{\lambda}_k = \frac{1}{\Lambda}\lambda_k.$$

For the associated eigenfunctions of  $H_{\tilde{\Omega}}$  we take

$$\tilde{\Phi}(x) := \Lambda^{-d/4}\Phi(x/\sqrt{\Lambda}) \quad (x \in \tilde{\Omega}),$$

so that, in particular,

$$\|\tilde{\Phi}\|_{L_2(\tilde{\Omega})} = \|\Phi\|_{L_2(\Omega)} \quad \text{and} \quad \|\tilde{\Phi}\|_{L_1(\tilde{\Omega})} = \Lambda^{d/4}\|\Phi\|_{L_1(\Omega)}.$$

Setting  $\tilde{r} := r/\Lambda$ ,  $\tilde{t} := t/\Lambda$  and using the estimate (1.4) of Theorem 1.2 for  $\tilde{\Phi}$  we obtain

$$\begin{aligned} \|\Phi\|_{L_1(\Omega)} &= \Lambda^{-d/4}\|\tilde{\Phi}\|_{L_1(\tilde{\Omega})} \\ &\leq C\Lambda^{-d/4} \left( \left( \frac{\tilde{r}^2}{\tilde{t}-\tilde{r}} \right)^{d/2} (\log N_{\tilde{t}}(H_{\tilde{\Omega}}))^{d/2} \sqrt{N_{\tilde{t}}(H_{\tilde{\Omega}})} + \tilde{r}^{-3/2} \left( \frac{\tilde{r}^2}{\tilde{t}-\tilde{r}} \right)^{2d} \right) \|\tilde{\Phi}\|_{L_2(\tilde{\Omega})}, \end{aligned}$$

and the desired result follows since  $N_{\tilde{t}}(H_{\tilde{\Omega}}) = N_t(H_{\Omega})$ . ■

From Theorem 1.3 we immediately get bounds on  $\|\Phi\|_p$  for any  $p \in [1, 2]$  as, trivially,  $\|\Phi\|_p^p \leq \|\Phi\|_1 + \|\Phi\|_2^2$ . A finer estimate is obtained through the inequality  $\|\Phi\|_p \leq \|\Phi\|_1^{\frac{2}{p}-1} \|\Phi\|_2^{2(1-\frac{1}{p})}$ .

In the special case where  $\sigma_{\text{ess}}(H_{\Omega}) = \emptyset$ , we may take  $\Lambda := \lambda_1$ ,  $r := \lambda_k$  and  $t := (1 + \vartheta)r = (1 + \vartheta)\lambda_k$ , with  $0 < \vartheta \leq 1$ , in Theorem 1.3, which gives the following.

**1.4 Corollary.** *For any  $d \in \mathbb{N}$  there exists a constant  $C \geq 0$  such that the following holds: If  $\Omega \neq \emptyset$  is an open subset of  $\mathbb{R}^d$  with  $\sigma_{\text{ess}}(H_{\Omega}) = \emptyset$ , we have*

$$\|\Phi\|_1^2 \leq C\lambda_1^{-d/2} \left( \vartheta^{-d} \left( \frac{\lambda_k}{\lambda_1} \right)^d (\log N_{(1+\vartheta)\lambda_k})^d N_{(1+\vartheta)\lambda_k} + \vartheta^{-4d} \left( \frac{\lambda_k}{\lambda_1} \right)^{4d-3} \right) \|\Phi\|_2^2,$$

for all  $0 < \vartheta \leq 1$  and for all eigenfunctions  $\Phi$  of  $H_{\Omega}$  associated with the eigenvalue  $\lambda_k$ .

The estimate given above will be applied in Section 5 to obtain a bound for the heat content  $Q_{\Omega}$  in terms of the heat trace  $Z_{\Omega}$ .

In many cases one will be satisfied with the choice  $\vartheta := 1$ , while smaller  $\vartheta$  may be of interest if  $N_t$  is of fast growth. Small  $\vartheta > 0$  are also important if one is interested in an estimate which depends on the gap length  $\lambda_{k+1} - \lambda_k$  (if  $\lambda_{k+1} > \lambda_k$ ); choosing  $\vartheta > 0$  so small that  $(1 + \vartheta)\lambda_k < \lambda_{k+1}$  we get  $N_{(1+\vartheta)\lambda_k} = N_{\lambda_k} = k$ .

**1.5 Remark.** In the special case  $d = 1$  one can obtain a sharper estimate by direct calculation, and it is instructive to do that. Any open set  $\Omega \subseteq \mathbb{R}$  can be written as a countable union of pairwise disjoint open intervals  $I_k \neq \emptyset$ . If one of these intervals has infinite length, we have  $\sigma_{\text{ess}}(H_\Omega) = [0, \infty)$  and we thus assume that all  $I_k$  have finite length  $\ell_k$ . In this case the operator  $H_\Omega$  has pure point spectrum and there is an orthonormal basis of eigenfunctions, each having support  $I_k$  for some  $k$ . Furthermore,  $\inf \sigma(H_\Omega) = \pi^2 \inf_k 1/\ell_k^2$  and  $\inf \sigma_{\text{ess}}(H_\Omega) = \pi^2 \liminf_{k \rightarrow \infty} 1/\ell_k^2$ . Assume that  $\inf \sigma(H_\Omega) < \inf \sigma_{\text{ess}}(H_\Omega)$ , and let  $\lambda \in [\inf \sigma(H_\Omega), \inf \sigma_{\text{ess}}(H_\Omega))$  be an eigenvalue of  $H_\Omega$ . Then there is a finite subset of the intervals  $I_k$ , which we may denote as  $I_1, \dots, I_K$  for simplicity, with the property that  $\lambda$  is an eigenvalue of  $H_{I_k}$ . This means that for each  $k = 1, \dots, K$  there is a number  $j_k \in \mathbb{N}$  such that

$$\lambda = \frac{\pi^2 j_k^2}{\ell_k^2}; \quad (1.7)$$

the associated normalized eigenfunction of  $H_{I_k}$  is given by

$$\varphi_k(x) := \sqrt{2/\ell_k} \sin \frac{\pi j_k}{\ell_k} (x - a_k),$$

if  $I_k = (a_k, b_k)$ . Here  $\|\varphi_k\|_1 = \frac{2\sqrt{2}}{\pi} \sqrt{\ell_k}$ .

Any eigenfunction  $\Phi$  of  $L_2$ -norm 1 of  $H_\Omega$  associated with the eigenvalue  $\lambda$  can be written as  $\Phi = \sum_{k=1}^K \alpha_k \varphi_k$  with  $\alpha_k \in \mathbb{C}$  and  $\sum_{k=1}^K |\alpha_k|^2 = 1$ . As for the  $L_1$ -norm of  $\Phi$ , we now estimate

$$\|\Phi\|_1 = \sum_k |\alpha_k| \|\varphi_k\|_1 = \frac{2\sqrt{2}}{\pi} \sum_k |\alpha_k| \sqrt{\ell_k} \leq \frac{2\sqrt{2}}{\pi} \left( \sum_k \ell_k \right)^{1/2},$$

by the Schwarz inequality. From (1.7) we get

$$\ell_k = \frac{\pi j_k}{\sqrt{\lambda}} = \frac{\pi}{\sqrt{\lambda}} N_\lambda(H_{I_k}),$$

whence  $\sum_k \ell_k \leq \frac{\pi}{\sqrt{\lambda}} N_\lambda(H_\Omega)$ . This leads to the estimate

$$\|\Phi\|_1^2 \leq C \lambda^{-1/2} N_\lambda(H_\Omega) \|\Phi\|_2^2,$$

with  $C = \frac{8}{\pi}$ . Comparing this last estimate with Theorem 0.1, we see that the leading power  $-d/2$  of the eigenvalue and a factor  $N_\lambda(H_\Omega)$  are there. However, the one-dimensional estimate above is stronger than the estimate given in Theorem 0.1, which was to be expected because there is no coupling between the  $H_{I_k}$ .

The following theorem gives an upper bound on the  $L_\infty$ -norm and a lower bound for the  $L_1$ -norm of the eigenfunctions of the Dirichlet Laplacian. It is a direct consequence of the domain monotonicity of the heat kernel ([22], [8; Theorem 2.1.6], [24; Theorem B.2]) and corresponding heat kernel bounds. Note that these bounds are valid for *all* eigenvalues and eigenfunctions. We include this well-known material chiefly for the sake of completeness.

**1.6 Theorem.** *Let  $\Omega \subseteq \mathbb{R}^d$  be open, and suppose that  $\lambda \in (0, \infty)$  is an eigenvalue of  $H_\Omega$ . Then*

$$\|\Phi\|_\infty \leq \left(\frac{e}{2\pi d}\right)^{d/4} \lambda^{d/4} \|\Phi\|_2, \quad (1.8)$$

and

$$\|\Phi\|_1 \geq \left(\frac{2\pi d}{e}\right)^{d/4} \lambda^{-d/4} \|\Phi\|_2, \quad (1.9)$$

for any eigenfunction  $\Phi$  of  $H_\Omega$  associated with the eigenvalue  $\lambda$ .

We defer the proof to the end of Section 5 where we will work with heat kernel estimates anyway.

We next look at the case where  $\sigma_{\text{ess}}(H_\Omega) \neq \emptyset$ . Choosing  $\Lambda \leq r \in [\Sigma_\Omega/4, \Sigma_\Omega)$  and letting  $t := (r + 2\Sigma_\Omega)/3$  in Theorem 1.3, we obtain estimates that display the dependence on the distance between  $\lambda_k$  and  $\Sigma_\Omega$ .

**1.7 Corollary.** *For any  $d \in \mathbb{N}$  there exists a constant  $C \geq 0$  such that the following holds: If  $\Omega \neq \emptyset$  is an open set in  $\mathbb{R}^d$  with  $\sigma_{\text{ess}}(H_\Omega) \neq \emptyset$ , and if  $r \in [\max\{\Lambda, \Sigma_\Omega/4\}, \Sigma_\Omega)$ , we have, writing also  $t_r := (r + 2\Sigma_\Omega)/3$ ,*

$$\begin{aligned} \|\Phi\|_1^2 \leq & C\Lambda^{-d/2} \left( \left( \frac{\Sigma_\Omega^2}{\Lambda(\Sigma_\Omega - r)} \right)^d (\log N_{t_r})^d N_{t_r} \right. \\ & \left. + \left( \frac{\Sigma_\Omega}{\Lambda} \right)^{-3} \left( \frac{\Sigma_\Omega^2}{\Lambda} (\Sigma_\Omega - r) \right)^{4d} \right) \|\Phi_k\|_2^2, \end{aligned}$$

for all eigenfunctions  $\Phi$  of  $H_\Omega$  associated with an eigenvalue  $\lambda_k \in [\Lambda, r]$ .

The above estimate is mainly of interest for eigenvalues  $\lambda_k$  close to  $\Sigma_\Omega$ . For eigenvalues  $\lambda_k$  close to  $\Lambda$ , a better, but also more complicated, estimate would be obtained by choosing  $r \in [\Lambda, \Sigma_\Omega)$  and  $t := \min\{3r, (r + 2\Sigma_\Omega)/3\}$ .

The following examples illustrate various points made in the preceding text.

**1.8 Examples.** (1) There are domains  $\Omega \subseteq \mathbb{R}^d$  such that  $H_\Omega$  has compact resolvent while  $\mathbb{R}^d \setminus \Omega$  has measure zero. In fact, consider a sequence of pairwise disjoint open cubes  $Q_k \subseteq \mathbb{R}^d$ ,  $k \in \mathbb{N}$ , enjoying the properties

- (i)  $\text{diam}(Q_k) \rightarrow 0$ , as  $k \rightarrow \infty$ ;
- (ii)  $\bigcup_{k \in \mathbb{N}} \overline{Q_k} = \mathbb{R}^d$ .

Then the Dirichlet Laplacian of  $\Omega := \bigcup_{k \in \mathbb{N}} Q_k$  has compact resolvent. In addition to properties (i) and (ii) one may require that any compact subset  $K \subseteq \mathbb{R}^d$  meets only finitely many of the  $Q_k$ .

To obtain a connected  $\Omega'$  from the above  $\Omega$  it is enough to open small “doors” in the surfaces that separate the cubes.

(2) Here we discuss examples of eigenfunctions which are not in  $L_1$ . Let  $\Omega = \bigcup_1^\infty I_k \subseteq \mathbb{R}$  be the disjoint union of open intervals of length 1,  $\varphi_0$  the normalized

eigenfunction of  $H_{(0,1)}$  to the lowest eigenvalue  $\lambda_0$ . Then  $\lambda_0 \in \sigma_{\text{ess}}(H_\Omega)$ . Let  $\varphi_k$  be the translate of  $\varphi_0$  to  $I_k$ , and let  $\alpha \in \ell_2 \setminus \ell_1$ . Then

$$\varphi := \sum_k \alpha_k \varphi_k \in L_2(\Omega)$$

is an eigenfunction of  $H_\Omega$  to the eigenvalue  $\lambda_0$ , but  $\varphi \notin L_1(\Omega)$ .

It is easy to generalize this idea to higher dimensions. Finding examples of domains  $\Omega$  in  $\mathbb{R}^d$ , for  $d \geq 2$ , with the property that  $H_\Omega$  has an eigenfunction which is not in  $L_1$  seems to be much harder. One might think of a quantum wave guide perturbed in such a way that an eigenvalue is generated right at a boundary point of the essential spectrum.

For the sake of comparison we note that there are examples of Schrödinger eigenfunctions on  $(0, \infty)$  which are not in  $L_1$  (see [12]); the associated eigenvalues belong to the essential spectrum.

(3) It is illuminating to compare the situation of  $m$  disjoint balls with the case where  $m$  balls are connected by thin passages, as in a dumb-bell domain for  $m = 2$ . Here one can see several aspects of the presence of  $N_{2\lambda_k}(H_\Omega)$  in our estimates.

We begin with a (disconnected) open set  $\Omega_m \subseteq \mathbb{R}^d$  consisting of  $m$  pairwise disjoint open balls of radius 1, say. Let  $\lambda_1$  denote the lowest eigenvalue of the Dirichlet Laplacian on such a ball. Then the lowest eigenvalue of  $H_{\Omega_m}$  is  $\lambda_1$  and the associated eigenspace has dimension  $m$ . It is easy to see that there is an eigenfunction  $\Phi$  of  $H_{\Omega_m}$  to the eigenvalue  $\lambda_1$  with the properties  $\|\Phi\|_2 = 1$  and  $\|\Phi\|_1^2 = m$ . Here an estimate involving  $N_{\lambda_1}(H_{\Omega_m})$  (instead of  $N_{2\lambda_1}(H_{\Omega_m})$ ) would be possible.

We now generalize domains of dumb-bell type. For  $2 \leq m \in \mathbb{N}$ , we place  $m$  balls of radius 1 at the corners of a regular  $m$ -gon with edge length 3, and connect each of these balls with its two neighbors by narrow passages of width  $0 < \varepsilon \leq 1$  along the edges. Call these domains  $\Omega_{m,\varepsilon}$ . By Perron-Frobenius theory, the ground state eigenvalue  $\lambda_{1;m,\varepsilon}$  of  $\Omega_{m,\varepsilon}$  is simple and the associated eigenfunction  $\Phi_{1;m,\varepsilon}$  can be chosen strictly positive; furthermore,  $\Phi_{1;m,\varepsilon}$  is invariant under rotation of the corners. Let  $\|\Phi_{1;m,\varepsilon}\|_2 = 1$ . As  $\varepsilon \downarrow 0$ , monotone convergence of quadratic forms, combined with compactness, implies that  $\|\Phi_{1;m,\varepsilon}\|_1^2 \rightarrow m$ .

In the disconnected case, we have  $N_{\lambda_1}(H_\Omega) = m$ , while  $N_{\lambda_1}(H_{\Omega_{m,\varepsilon}})$  is equal to one in the connected case. In the connected case, however, there is a cluster of  $m$  eigenvalues close to  $\lambda_1$  (for  $\varepsilon > 0$  small), and  $N_{2\lambda_1}(H_{\Omega_{m,\varepsilon}}) \rightarrow m$  as  $\varepsilon \downarrow 0$ . Therefore, in the connected case the estimate should better contain a factor like  $N_t(H_{\Omega_{m,\varepsilon}})$ , with suitable  $t > \lambda_1$ .

Note that, in both cases, the lower bound of Theorem 1.6 does not capture the above behavior since it provides a constant which is independent of  $m$ .

(4) Examples of open sets  $\Omega \subseteq \mathbb{R}^d$  with discrete eigenvalues located in a gap of the essential spectrum can be obtained by suitable perturbations of periodic quantum wave-guides. Consider open, connected, periodic sets  $\Omega_0 \subseteq \mathbb{R}^2$  of the form

$$\Omega_0 = \{(x, y) \in \mathbb{R}^2; f_1(x) < y < f_2(x)\}$$

where  $f_1$  and  $f_2$  are smooth periodic functions of the same period satisfying  $f_1(x) < f_2(x)$ . The spectrum of  $H_{\Omega_0}$  is pure essential spectrum. In this class it is easy to

find domains with a spectral gap. The simplest examples are obtained by joining discs  $B((k, 0), 1/4)$  ( $k \in \mathbb{Z}$ ) by narrow passages along the  $x$ -axis, but there are also more demanding examples like the ones studied by Yoshitomi [27]. Local perturbations of the boundary of  $\Omega_0$  may produce discrete eigenvalues below the essential spectrum, but also discrete eigenvalues inside a given gap of the essential spectrum. See for example [23].

(5) We finally discuss a class of examples which are closely related to Remark 1.5. Let  $\Omega_0 \subseteq \mathbb{R}^d$  be open and bounded. Let  $(\ell_k)_{k \in \mathbb{N}}$  be a sequence in  $(0, \infty)$ , and let  $\Omega$  be the disjoint union of a sequence  $(\Omega_k)$ , where  $\Omega_k$  is a translate of  $\ell_k \Omega_0$ , for all  $k \in \mathbb{N}$ . For a dilation  $\ell \Omega_0$ , with  $\ell > 0$ , it follows from the lower bound given in (1.10) below that

$$N_\lambda(H_{\ell \Omega_0}) = N_{\ell^2 \lambda}(H_{\Omega_0}) \geq c_0 \ell^d \lambda^{d/2}$$

for all eigenvalues  $\lambda$  of  $H_{\ell \Omega_0}$ , with a positive constant  $c_0$ .

Assume that  $\inf \sigma(H_\Omega) < \inf \sigma_{\text{ess}}(H_\Omega)$ , and let  $\lambda \in [\inf \sigma(H_\Omega), \inf \sigma_{\text{ess}}(H_\Omega))$  be an eigenvalue of  $H_\Omega$ , with associated eigenfunction  $\Phi$ . Arguing as in Remark 1.5, one then obtains that

$$\|\Phi\|_1^2 \leq \frac{\text{vol}(\Omega_0)}{c_0} \lambda^{-d/2} N_\lambda(H_\Omega) \|\Phi\|_2^2.$$

For completeness we include here a simple lower bound for the eigenvalue counting function  $N_\lambda(H_\Omega)$  which does not invoke Weyl's Theorem and which comes with an explicit constant.

Let  $\Omega$  be an open set in  $\mathbb{R}^d$  and suppose that  $\sigma_{\text{ess}}(H_\Omega) = \emptyset$ . Let  $\mathcal{C}(\Omega)$  be the collection of open cubes contained in  $\Omega$  and define

$$\gamma(\Omega) := \sup\{\text{vol}(A) ; A \in \mathcal{C}(\Omega)\}.$$

Let  $A \in \mathcal{C}(\Omega)$ ,  $\text{vol}(A) = a^d$ . By domain monotonicity of the Dirichlet eigenvalues we have that

$$\begin{aligned} N_t(H_\Omega) &\geq N_t(H_A) = |\{(k_1, \dots, k_d) \in \mathbb{N}^d ; \pi^2(k_1^2 + \dots + k_d^2) \leq ta^2\}| \\ &\geq |\{k \in \mathbb{N} ; d\pi^2 k^2 \leq ta^2\}|^d \geq \left[ \frac{at^{1/2}}{\pi d^{1/2}} \right]^d. \end{aligned}$$

Since  $\max\{[x], 1\} \geq x/2$ , we conclude that

$$N_t(H_\Omega) \geq \left( \frac{at^{1/2}}{2\pi d^{1/2}} \right)^d = (2\pi)^{-d} d^{-d/2} \text{vol}(A) t^{d/2} \quad (t \geq \lambda_1).$$

Taking the supremum over all  $A \in \mathcal{C}(\Omega)$  we finally obtain the lower bound

$$N_t(H_\Omega) \geq (2\pi)^{-d} d^{-d/2} \gamma(\Omega) t^{d/2} \quad (t \geq \lambda_1). \quad (1.10)$$

## 2 Proof of Proposition 1.1

We define coverings of  $\mathbb{R}^d$  by cubes  $Q_{n,j}$  ( $j \in \mathbb{Z}^d$ ), for  $n \in \mathbb{N}$ , and subordinate IMS-partitions of unity  $(\Psi_{n,j})_{j \in \mathbb{Z}^d}$ . Let  $Q_0 := (-1, 1)^d$  denote the standard cube of side length 2 centered at the point  $0 \in \mathbb{R}^d$ , and let  $Q_j := Q_0 + j$ , for  $j \in \mathbb{Z}^d$ , denote the translates of  $Q_0$ . Pick some non-negative function  $\psi \in C_c^\infty(Q_0)$ , with  $\psi(x) \geq 1$  for all  $x \in \frac{1}{2}Q_0$ . Let  $\psi_j \in C_c^\infty(Q_j)$  be defined by  $\psi_j(x) := \psi_0(x - j)$ . Extending the  $\psi_j$  by zero to all of  $\mathbb{R}^d$ , we note that the function

$$w := \sum_{j \in \mathbb{Z}^d} \psi_j^2$$

is periodic and positive. We now define the IMS-partition of unity  $(\Psi_j)_{j \in \mathbb{Z}^d}$  ([7]) by

$$\Psi_j := \frac{\psi_j}{\sqrt{w}} \quad (j \in \mathbb{Z}^d),$$

so that  $\sum_{j \in \mathbb{Z}^d} \Psi_j^2(x) = 1$  for all  $x \in \mathbb{R}^d$ . (Notice the square; this is *not* a standard partition of unity!) Obviously  $\text{spt } \Psi_j \subseteq Q_j$  for all  $j \in \mathbb{Z}^d$ . Furthermore,  $\Psi_j$  is a translate of  $\Psi_0$ , and thus

$$c := \|\nabla \Psi_0\|_\infty = \|\nabla \Psi_j\|_\infty \quad (j \in \mathbb{Z}^d). \quad (2.1)$$

(It would be easy to indicate an upper bound for  $c$  in terms of  $\|\nabla \psi\|_\infty$ .)

We finally produce scaled versions defined as

$$\Psi_{n,j} := \Psi_j\left(\frac{\cdot}{n}\right) \quad (n \in \mathbb{N}, j \in \mathbb{Z}^d);$$

notice that  $\Psi_{n,j}$  has support in the cube  $Q_{n,j} := nQ_j = nQ_0 + nj$ . Then

$$\sum_{j \in \mathbb{Z}^d} \Psi_{n,j}^2(x) = 1 \quad (x \in \mathbb{R}^d),$$

and

$$\|\nabla \Psi_{n,j}\|_\infty = c/n \quad (n \in \mathbb{N}, j \in \mathbb{Z}^d). \quad (2.2)$$

For  $1 < t < \inf \sigma_{\text{ess}}(H_\Omega)$  and  $n \in \mathbb{N}$ , we now let

$$J(n, t) := \{j \in \mathbb{Z}^d; \lambda_1(H_{\Omega \cap Q_{n,j}}) < t\}.$$

We then define

$$F_n := \bigcup_{j \in J(n, t)} Q_{n,j}.$$

For later use we also introduce

$$\tilde{Q}_0 := 2Q_0 = (-2, 2)^d, \quad \tilde{Q}_{n,j} := n\tilde{Q}_0 + nj, \quad \tilde{F}_n := \bigcup_{j \in J(n, t)} \tilde{Q}_{n,j},$$

$$\tilde{Q}_0 := 3Q_0 = (-3, 3)^d, \quad \tilde{Q}_{n,j} := n\tilde{Q}_0 + nj, \quad \tilde{F}_n := \bigcup_{j \in J(n,t)} \tilde{Q}_{n,j}.$$

We then have  $F_n \subseteq \tilde{F}_n \subseteq \tilde{F}_n$ , and

$$\text{dist}(F_n, \partial\tilde{F}_n) \geq n, \quad \text{dist}(\tilde{F}_n, \partial\tilde{F}_n) \geq n,$$

for all  $n \in \mathbb{N}$ .

The following lemma shows that there is only a finite number of “cells”  $Q_{n,j}$  such that the infimum of the spectrum of the Dirichlet Laplacian of  $\Omega \cap Q_{n,j}$  is smaller than  $t$ .

**2.1 Lemma.** *Let  $1 < t < \inf \sigma_{\text{ess}}(H_\Omega)$  and let  $n \in \mathbb{N}$ . Then  $J(n, t)$  is finite and*

$$|J(n, t)| \leq 3^d N_t(H_\Omega),$$

where  $|J(n, t)|$  denotes the number of elements in  $J(n, t)$ .

*Proof.* Let  $J \subseteq J(n, t)$  be finite. There exists  $J' \subseteq J$  such that the cubes  $(Q_{n,j})_{j \in J'}$  are pairwise disjoint and  $nJ \subseteq \bigcup_{j \in J'} \overline{Q_{n,j}}$ . The latter property implies that  $|J| \leq 3^d |J'|$ .

For each of the cubes  $Q_{n,j}$  ( $j \in J'$ ) there exists a function  $\varphi_j \in C_c^\infty(\Omega \cap Q_{n,j})$  with  $\|\varphi_j\| = 1$  and  $\|\nabla \varphi_j\|^2 < t$ . Then the min-max principle, applied to the subspace spanned by the set  $\{\varphi_j; j \in J'\}$ , implies that  $|J'| \leq N_t(H_\Omega)$ . The assertions follow from these two inequalities.  $\blacksquare$

Below we obtain a lower bound for the spectrum of the Dirichlet Laplacian in  $G_n := \Omega \setminus \overline{F_n}$ .

**2.2 Lemma.** *Let  $1 < s < t < \inf \sigma_{\text{ess}}(H_\Omega)$  and  $n \geq n_0 := 2^{d/2}c/\sqrt{t-s}$ , with  $c$  from (2.1). Let  $H_{G_n}$  denote the Dirichlet Laplacian of  $G_n := \Omega \setminus \overline{F_n}$ . Then*

$$\inf \sigma(H_{G_n}) \geq s.$$

*Proof.* By the IMS-localization formula [7; Theorem 3.2] and (2.2) we have for any  $\varphi \in C_c^\infty(G_n)$

$$\begin{aligned} \langle H_{G_n} \varphi, \varphi \rangle &= \sum_{j \in \mathbb{Z}^d} \langle H_{G_n} \Psi_{n,j} \varphi, \Psi_{n,j} \varphi \rangle - \int \sum_{j \in \mathbb{Z}^d} |\nabla \Psi_{n,j}|^2 |\varphi|^2 dx \\ &\geq t \sum_{j \in \mathbb{Z}^d} \|\Psi_{n,j} \varphi\|^2 - 2^d \frac{c^2}{n^2} \|\varphi\|^2 \\ &= (t - \frac{2^d c^2}{n^2}) \|\varphi\|^2 \geq s \|\varphi\|^2. \end{aligned}$$

In the estimate we have used that  $\Psi_{n,j} \varphi = 0$  for  $j \in J(n, t)$ . The factor  $2^d$  takes into account the fact that at most  $2^d$  functions  $\Psi_{n,j}$  can be simultaneously non-zero at any given point  $x$ .  $\blacksquare$

We next consider a smoothed version of the indicator function of  $\tilde{F}_n$ , defined as

$$\xi_n := \varrho * \mathbf{1}_{\tilde{F}_n}, \quad (2.3)$$

with  $\varrho$  defined in Section 1. For  $n \in \mathbb{N}$  we have  $\text{spt } \xi_n \subseteq \tilde{F}_n$  and  $\xi_n(x) = 1$  for  $x \in F_n$ . Furthermore,  $0 \leq \xi_n(x) \leq 1$  and  $\|\nabla \xi_n(x)\|_\infty \leq C$ ,  $\|\Delta \xi_n(x)\|_\infty \leq C$  for some constant  $C \geq 0$  which is independent of  $n$  and  $\Omega$ . Also,

$$\text{spt } \nabla \xi_n \subseteq \{x \in \mathbb{R}^d; \text{dist}(x, \partial \tilde{F}_n) < 1/2\}.$$

It will be convenient to cover the support of  $\nabla \xi_n$  by (non-overlapping) cubes of side length 1, given by

$$\check{Q}_0 := (-1/2, 1/2]^d, \quad \check{Q}_\ell := \check{Q}_0 + \ell \quad (\ell \in \mathbb{Z}^d), \quad (2.4)$$

and we will write  $\check{\chi}_\ell := \mathbf{1}_{\check{Q}_\ell}$ .

We then let

$$Z_n := \mathbb{Z}^d \cap \partial \tilde{F}_n. \quad (2.5)$$

Note that this implies  $\text{spt } \nabla \xi_n \subseteq \bigcup_{\ell \in Z_n} \check{Q}_\ell$ . We then have

$$|Z_n| \leq |J(n, t)| |\mathbb{Z}^d \cap \partial \tilde{Q}_{n,0}| \leq C n^{d-1} N_t(H_\Omega), \quad (2.6)$$

by Lemma 2.1, with  $C = 2^{3d-2} 3^d$ . We furthermore let

$$Y_n := \{j \in \mathbb{Z}^d; \check{Q}_j \cap (\Omega \setminus \tilde{F}_n) \neq \emptyset\},$$

so that  $\Omega \setminus \tilde{F}_n \subseteq \bigcup_{j \in Y_n} \check{Q}_j$ .

We now quantify the exponential decay of  $\Phi$  as we move away from the set  $\tilde{F}_n$ .

**2.3 Lemma.** *There exists a constant  $C \geq 0$  with the following property. If  $1 \leq r < t < \inf \sigma_{\text{ess}}(H_\Omega)$ , if  $n \geq n_0$  (with  $n_0$  from Lemma 2.2), and if  $\xi_n$  from (2.3), then*

$$\|\check{\chi}_j(1 - \xi_n)\Phi\|_1 \leq C \frac{\sqrt{r}}{t - r} \sum_{\ell \in Z_n} e^{-\alpha|j - \ell|} \quad (j \in \mathbb{Z}^d)$$

(with  $\alpha$  from (1.2)), for all normalized eigenfunctions  $\Phi$  associated with an eigenvalue  $\lambda_k \in [1, r]$ .

We defer the proof of Lemma 2.3 to Section 3.

*Proof of Proposition 1.1.* First we treat the case where  $n \in \mathbb{N}$ , starting from

$$\|\Phi\|_1 = \int_{\Omega \cap \tilde{F}_n} |\Phi(x)| dx + \int_{\Omega \setminus \tilde{F}_n} |\Phi(x)| dx =: I_{n,1} + I_{n,2}. \quad (2.7)$$



By the Schwarz inequality and Lemma 2.1, the first term on the right hand side of (2.7) can be estimated as follows:

$$I_{n,1} \leq \text{vol}_d(\tilde{F}_n)^{1/2} \|\Phi\|_2 \leq |J(n, t)|^{1/2} (6n)^{d/2} \leq 3^d 2^{d/2} n^{d/2} \sqrt{N_t(H_\Omega)}. \quad (2.8)$$

As for the second term on the right hand side of (2.7), we note that for  $j \in Y_n$  and  $\ell \in Z_n$  we have

$$|j - \ell| \geq n - 1.$$

It now follows by Lemma 2.3 that

$$\begin{aligned} I_{n,2} &\leq \sum_{j \in Y_n} \|\check{\chi}_j(1 - \xi_n)\Phi\|_1 \leq C \frac{\sqrt{r}}{t - r} \sum_{j \in Y_n} \sum_{\ell \in Z_n} e^{-\alpha|j - \ell|} \\ &= C \frac{\sqrt{r}}{t - r} \sum_{\ell \in Z_n} \sum_{j \in Y_n} e^{-\alpha|j - \ell|} \leq C \frac{\sqrt{r}}{t - r} \cdot \left( \sup_{\ell \in Z_n} M(n, \ell) \right) \cdot |Z_n|, \end{aligned}$$

where  $M(n, \ell) := \sum_{j \in Y_n} e^{-\alpha|j - \ell|}$  for  $\ell \in Z_n$ . Here  $|Z_n| \leq C n^{d-1} N_t(H_\Omega)$  by (2.6), and

$$\begin{aligned} M(n, \ell) &\leq \sum_{j \in \mathbb{Z}^d, |j| \geq n-1} e^{-\alpha|j|} \leq e^{\alpha\sqrt{d}} \int_{\{\xi \in \mathbb{R}^d; |\xi| \geq n-1\}} e^{-\alpha|\xi|} d\xi \\ &\leq C \left( \frac{n^{d-1}}{\alpha} + \frac{1}{\alpha^d} \right) e^{-\alpha n}, \end{aligned}$$

for all  $n \in \mathbb{N}$  and  $\ell \in Z_n$ . We therefore obtain the estimate

$$I_{n,2} \leq C \frac{\sqrt{r}}{t - r} n^{d-1} N_t(H_\Omega) \left( \frac{n^{d-1}}{\alpha} + \frac{1}{\alpha^d} \right) e^{-\alpha n}. \quad (2.9)$$

Now (1.3) follows from (2.8) and (2.9).

Finally, we reduce the case of non-integer  $n$  to the case treated above. If  $n \in (0, \infty)$  satisfies the required inequality, then  $\tilde{n} := \lceil n \rceil$  (the smallest integer  $\geq n$ ) belongs to  $\mathbb{N}$ , and the asserted inequality holds for  $n$  replaced with  $\tilde{n}$ . Readjusting the constant  $C$ , one then obtains the estimate with  $n$ .  $\blacksquare$

### 3 Exponential decay estimates

This section is devoted to quantitative exponential decay estimates and a proof of Lemma 2.3. We use the method of *boosting*, a well-known tool in the study of eigenfunctions of Schrödinger operators (cf. [11] or [18] for a survey of the literature). Here the operator is sandwiched between  $e^{\gamma \cdot x}$  and  $e^{-\gamma \cdot x}$  for  $\gamma \in \mathbb{R}^d$ . This method dates from the early seventies and yields exponential decay in the  $L_2$ -sense. Below, we follow to some extent the proof of [10; Lemma 6]. To avoid discussing operator domains we will not work with  $e^{\pm \gamma \cdot x}$  directly but rather with smoothed cut-offs of these functions.

We first consider general real-valued functions  $f \in C^\infty(\mathbb{R}^d)$  with  $f$ ,  $\nabla f$  and  $\Delta f$  bounded; only later on we will specify  $f$  to coincide with  $\gamma \cdot (x - k)$  on a large ball. Let  $G \subseteq \mathbb{R}^d$  open. By [19; Theorem VI-2.1], it is then easy to see that  $f u \in D(H_G)$  for all  $u \in D(H_G)$ ; furthermore,

$$e^{-f} H_G e^f = H_G - 2\nabla f \cdot \nabla - \Delta f - |\nabla f|^2.$$

Here we note that the perturbation  $-2\nabla f \cdot \nabla - \Delta f - |\nabla f|^2$  has relative form-bound zero with respect to  $H_G$  on  $H_0^1(G)$ . We let  $H_{G,f}$  denote the (unique)  $m$ -sectorial closed operator associated with  $H_G - 2\nabla f \cdot \nabla - \Delta f - |\nabla f|^2$  by [19; Theorem VI-3.4 or 3.9]. Also, using [19; Theorem VI-2.1(iii)], one can easily see that  $D(H_{G,f}) = D(H_G)$ . From the inequalities

$$\|\nabla \varphi\|_2^2 = h_G[\varphi, \varphi] = \langle H_G \varphi, \varphi \rangle \leq \frac{\delta^2}{2} \|H_G \varphi\|_2^2 + \frac{1}{2\delta^2} \|\varphi\|_2^2, \quad (3.1)$$

valid for all  $\varphi \in D(H_G)$  and all  $\delta > 0$ , we see that  $H_{G,f} - H_G$  is relatively bounded with respect to  $H_G$  in the operator sense, with relative bound zero.

**3.1 Lemma.** *Let  $G \subseteq \mathbb{R}^d$  be open and let  $H_G$ , the Dirichlet Laplacian of  $G$ , be such that  $\sigma_0 := \inf \sigma(H_G) > 1$ . For  $1 \leq r < s < \sigma_0$  and  $m \geq 1$  define*

$$\alpha := \frac{\min\{s - r, 1\}}{16mr}. \quad (3.2)$$

*Let  $f \in C^\infty(\mathbb{R}^d; \mathbb{R})$  be bounded with  $\|\nabla f\|_\infty \leq m$  and  $\|\Delta f\|_\infty \leq m$ .*

*We then have  $[1, r] \subseteq \varrho(H_{G,\alpha f})$ , and*

$$\|(H_{G,\alpha f} - \lambda)^{-1}\| \leq \frac{2}{s - r},$$

*for all  $\lambda \in [1, r]$ . Furthermore, for the same  $\lambda$ , one has*

$$(H_{G,\alpha f} - \lambda)^{-1} = e^{-\alpha f} (H_G - \lambda)^{-1} e^{\alpha f}.$$

*Proof.* We are going to apply [19; Theorem IV-1.16] to  $T := H_G - \lambda$  and  $S := H_{G,\alpha f} - \lambda$ . In estimating the term containing  $\nabla f \cdot \nabla$  we use (3.1) with  $\delta := 1$  so that

$$2\|\nabla f \cdot \nabla \varphi\|_2 \leq \sqrt{2}m\|(H_G - \lambda)\varphi\|_2 + \sqrt{2}m(\lambda + 1)\|\varphi\|_2.$$

It is not difficult to see that the numbers  $a := \sqrt{2}m|\alpha|(2 + \lambda) + |\alpha|^2 m^2$  and  $b := \sqrt{2}|\alpha|m$  satisfy the condition  $a\|(H_G - \lambda)^{-1}\| + b \leq 1/2$ . We thus see that any  $\lambda \in [1, r]$  belongs to the resolvent set of  $H_{G,\alpha f}$ ; furthermore, [19; equation IV-(1.31)] yields the estimate  $\|(H_{G,\alpha f} - \lambda)^{-1}\| \leq \frac{2}{s-r}$ . Direct computation shows that  $e^{-\alpha f}(H_G - \lambda)^{-1}e^{\alpha f}$  is the inverse of  $H_{G,\alpha f} - \lambda$ .  $\blacksquare$

For the application of Lemma 3.1 we need to construct specific functions  $f$ . Consider first

$$\varphi_{k,\ell}(x) := \frac{1}{|\ell - k|} \langle x - k, \ell - k \rangle \quad (x \in \mathbb{R}^d, \ k, \ell \in \mathbb{Z}^d, \ k \neq \ell),$$

so that  $\varphi_{k,\ell}(k) = 0$  and  $\varphi_{k,\ell}(\ell) = |\ell - k|$ . We next take, for  $R \geq 1$ ,

$$f_{R,k,\ell} := \varrho_R * ((\varphi_{k,\ell} \wedge R) \vee (-R)),$$

with  $\varrho$  defined in Section 1 and  $\varrho_R := \frac{1}{R^d} \varrho(\frac{\cdot}{R})$ ; recall that  $\text{spt } \varrho \subseteq B(0, 1/2)$ . We then have  $\|\nabla f_{R,k,\ell}\|_\infty \leq 1$  and  $\|\Delta f_{R,k,\ell}\|_\infty \leq \frac{1}{R} \|\Delta \varrho\|_1$ .

**3.2 Lemma.** *Let  $G \subseteq \mathbb{R}^d$  be an open set with  $\sigma_0 := \inf \sigma(H_G) > 1$ , and let  $1 \leq r < s < \sigma_0$ . Let*

$$\alpha := \frac{\min\{s - r, 1\}}{16m_0r},$$

with  $m_0$  from (1.1). Finally, let  $\check{Q}_k$  as in (2.4), and  $\check{\chi}_k := \mathbf{1}_{\check{Q}_k}$  for  $k \in \mathbb{Z}^d$ .

Then there exists a constant  $C \geq 0$ , depending only on  $d$ , such that

$$\|\check{\chi}_k(H_G - \lambda)^{-1} \check{\chi}_\ell\| \leq \frac{C}{s - r} e^{-\alpha|k - \ell|} \quad (k, \ell \in \mathbb{Z}^d),$$

for any  $\lambda \in [1, r]$ .

*Proof.* In the cases where  $k = \ell$ , the inequality holds with  $C = 1$ .

Let  $k, \ell \in \mathbb{Z}^d$ ,  $k \neq \ell$ , and let  $R := 2|k - \ell|$ . With  $f := \alpha f_{R,k,\ell}$  and  $E_f$  denoting multiplication by the function  $e^f$ , we compute

$$\begin{aligned} \|\check{\chi}_k(H_G - \lambda)^{-1} \check{\chi}_\ell\| &= \|\check{\chi}_k E_f E_{-f} (H_G - \lambda)^{-1} E_f E_{-f} \check{\chi}_\ell\| \\ &\leq \|\check{\chi}_k e^f\|_\infty \|E_{-f} (H_G - \lambda)^{-1} E_f\| \|e^{-f} \check{\chi}_\ell\|_\infty \\ &\leq \frac{C_0}{s - r} \|e^{-f} \check{\chi}_\ell\|_\infty, \end{aligned}$$

by Lemma 3.1. Since  $0 < \alpha \leq 1$  we have  $\|\check{\chi}_k e^f\|_\infty \leq e^{\sqrt{d}/2}$  and we may thus choose  $C_0 := 9e^{\sqrt{d}/2}$ . Furthermore,

$$\|e^{-f} \check{\chi}_\ell\|_\infty \leq \sup_{x \in \check{Q}_\ell} e^{-\alpha \varphi_{k,\ell}(x)} \leq e^{-\alpha(|\ell - k| - \sqrt{d}/2)} = C_1 e^{-\alpha|k - \ell|},$$

with  $C_1 := e^{\alpha\sqrt{d}/2}$ . ■

*Proof of Lemma 2.3.* In this proof we specify the number  $s$ , occurring in Lemma 2.2 and in Lemma 3.2, as

$$s := \frac{r + t}{2}.$$

With this definition the quantity  $\alpha$  occurring in Lemma 3.2 becomes  $\alpha$  from (1.2), and  $n_0$  from Lemma 2.2 becomes the lower bound for  $n$  in Proposition 1.1.

Let  $n \geq n_0$  (from Lemma 2.2), and let  $\xi_n \in C_c^\infty(\mathbb{R}^d)$  be as defined in (2.3). As in Lemma 2.2 we consider  $G = G_n := \Omega \setminus \overline{F_n}$ , and we conclude from Lemma 2.2 that  $\inf \sigma(H_G) \geq s$ , for the Dirichlet Laplacian  $H_G$  of  $G$ .

Let  $\Phi$  be a normalized eigenfunction of  $H_\Omega$ , associated with an eigenvalue  $\lambda_k \in [1, r]$ . It is easy to see that  $(1 - \xi_n)\Phi$  belongs to  $D(H_\Omega) \cap D(H_G)$ . The usual calculation yields

$$\begin{aligned} (H_G - \lambda_k)((1 - \xi_n)\Phi) &= (H_\Omega - \lambda_k)((1 - \xi_n)\Phi) \\ &= 2\nabla\xi_n \cdot \nabla\Phi + (\Delta\xi_n)\Phi =: \eta. \end{aligned} \quad (3.3)$$

Here  $\text{spt } \eta \subseteq \text{spt } \nabla\xi_n \subseteq \bigcup_{\ell \in Z_n} \tilde{Q}_\ell$ , with  $Z_n$  from (2.5), and

$$\|\eta\|_2 \leq C(1 + \|\nabla\Phi\|_2) = C(1 + \sqrt{\lambda_k}),$$

with a constant  $C \geq 0$  depending only on  $\|\nabla\xi_n\|_\infty$  and  $\|\Delta\xi_n\|_\infty$  (and therefore not depending on  $n$  and  $\Omega$ ), and (3.3) yields

$$(1 - \xi_n)\Phi = (H_G - \lambda_k)^{-1}\eta. \quad (3.4)$$

We now obtain

$$\check{\chi}_j(1 - \xi_n)\Phi = \check{\chi}_j(H_G - \lambda)^{-1} \left( \sum_{\ell \in Z_n} \check{\chi}_\ell \eta \right)$$

for any  $j \in \mathbb{Z}^d$ , so that, by Lemma 3.2 and (3.4),

$$\begin{aligned} \|\check{\chi}_j(1 - \xi_n)\Phi\|_2 &\leq \sum_{\ell \in Z_n} \|\check{\chi}_j(H_G - \lambda)^{-1} \check{\chi}_\ell\| \|\eta\|_2 \\ &\leq \frac{C}{t-r} \sum_{\ell \in Z_n} e^{-\alpha|j-\ell|} (1 + \sqrt{\lambda_k}) \\ &\leq C \frac{\sqrt{r}}{t-r} \sum_{\ell \in Z_n} e^{-\alpha|j-\ell|}. \end{aligned}$$

From Hölder's inequality we obtain  $\|\check{\chi}_j(1 - \xi_n)\Phi\|_1 \leq \|\check{\chi}_j(1 - \xi_n)\Phi\|_2$ , and this completes the proof.  $\blacksquare$

## 4 A general result for discrete eigenvalues

Here we show that it is a very general property that eigenfunctions of the Dirichlet Laplacian corresponding to isolated eigenvalues of finite multiplicity are integrable. This is more general than what is proved in the previous sections and applies also to eigenvalues above the infimum of the essential spectrum, lying in gaps of the essential spectrum. However, this general result does not give information about  $L_1$ -bounds of the eigenfunctions, which is the most important point in the previous sections (and in fact in the present paper).

It would be of interest to obtain  $L_1$ -estimates for eigenfunctions associated with a discrete eigenvalue located in a gap of  $\sigma_{\text{ess}}(H_\Omega)$  above the infimum of the essential spectrum.

Let  $H (= -\Delta)$  denote the Laplacian in  $L_2(\mathbb{R}^d)$ , and let  $\Omega \subseteq \mathbb{R}^d$  be open. Then the  $C_0$ -semigroup generated by  $-H_\Omega$  is dominated by the  $C_0$ -semigroup generated by  $-H$ ; see e.g. [22], [8; Theorem 2.1.6], [24; Theorem B.2]. This implies that the  $C_0$ -semigroup generated by  $-H_\Omega$  is associated with an integral kernel satisfying a Gaussian estimate.

**4.1 Theorem.** *Let  $\Omega \subseteq \mathbb{R}^d$  be an open set, and assume that  $\lambda$  is an eigenvalue of  $H_\Omega$  of finite algebraic multiplicity (or in other words,  $\lambda$  is an isolated point of the spectrum of  $H_\Omega$  and an eigenvalue of  $H_\Omega$  of finite multiplicity). Then the eigenspace corresponding to  $\lambda$  is a subspace of  $L_p(\Omega)$ , for all  $p \in [1, \infty]$ .*

*Proof.* For  $2 \leq p \leq \infty$ , the assertion follows from the facts that the eigenspace corresponding to  $\lambda$  is invariant under the  $C_0$ -semigroup generated by  $H_\Omega$  and that the Gaussian estimate of the semigroup kernel implies that  $L_2(\Omega)$  is mapped to  $L_p(\Omega)$  for positive times ( $p$ - $q$ -smoothing property of the semigroup for  $1 \leq p \leq q \leq \infty$ ). For  $1 \leq p < 2$  we recall from [1; Corollary 4.3 and Example 5.1(a)] that the component  $\varrho_\infty(-H_{\Omega,p})$  of the  $L_p$ -resolvent set of  $-H_\Omega$  containing the right half-plane (which for  $p = 2$  is equal to the resolvent set of  $H_\Omega$ , because the spectrum is a subset of  $(-\infty, 0]$ ) is independent of  $1 \leq p < \infty$ . Moreover, it is shown in [1] that the resolvents are consistent in  $\varrho(H_\Omega)$ .

Now, the hypothesis states that  $\lambda$  is a pole of the resolvent of  $H_\Omega$ , with finite rank residuum (which is just the corresponding spectral projection). Then we conclude from [17; Theorem 1.3] (see also [2]) that  $\lambda$  is an eigenvalue of finite algebraic multiplicity of  $H_{\Omega,p}$  (where  $-H_{\Omega,p}$  denotes the generator of the  $L_p$ -semigroup), for all  $1 \leq p < \infty$ , and that the range of the residuum is independent of  $p$ . As the eigenspace corresponding to the eigenvalue  $\lambda$  is just the range of the residuum we conclude that it is a subspace of  $L_p(\Omega)$  for all  $p \in [1, \infty)$ . ■

## 5 Heat content and heat trace

We let  $(e^{-tH_\Omega}; t \geq 0)$  denote the  $C_0$ -semigroup generated by  $H_\Omega$  in  $L_2(\Omega)$ . For  $f$  continuous and bounded,  $(e^{-tH_\Omega} f; t \geq 0)$  provides a (weak) solution of the initial boundary value problem for the heat equation, given by

$$\frac{\partial u}{\partial t} = \Delta u \quad (x \in \Omega, t > 0),$$

where  $\lim_{t \downarrow 0} u(\cdot; t) = f$ , locally uniformly, and  $u(\cdot; t) = 0$  on  $\partial\Omega$  for  $t > 0$  in the usual weak sense that  $u(\cdot; t) \in H_0^1(\Omega)$ . As is well-known [15], there is a smooth function

$$\Omega \times \Omega \times (0, \infty) \ni (x, y; t) \mapsto p_\Omega(x, y; t),$$

called the *Dirichlet heat kernel for  $\Omega$* , such that

$$(e^{-tH_\Omega} f)(x) = \int_\Omega p_\Omega(x, y; t) f(y) dy \quad (x \in \Omega, t > 0).$$

In particular

$$u(x; t) := \int_\Omega p_\Omega(x, y; t) dy \quad (x \in \Omega, t > 0)$$

solves the above initial boundary value problem for the constant function  $f = 1$ . At regular boundary points  $x_0 \in \partial\Omega$  we have  $u(x; t) \rightarrow 0$  as  $\Omega \ni x \rightarrow x_0$ , for any  $t > 0$ . Physically,  $u(x; t)$  represents the temperature at a point  $x$  at time  $t$  if  $\Omega$  initially has constant temperature 1, while the boundary is kept at temperature 0 for all  $t > 0$ .

The *heat content* of  $\Omega$  at time  $t > 0$  is defined by

$$Q_\Omega(t) := \int_\Omega u(x; t) \, dx = \int_\Omega \int_\Omega p_\Omega(x, y; t) \, dy \, dx \quad (t > 0).$$

This quantity has been studied extensively in the general setting of open bounded sets with smooth boundaries in complete Riemannian manifolds. See for example [6, 14].

For  $H_\Omega$  with compact resolvent, we let  $(\Phi_k)_{k \in \mathbb{N}}$  denote an orthonormal basis of real eigenfunctions of  $H_\Omega$ , associated with the increasing sequence of eigenvalues  $(\lambda_k)_{k \in \mathbb{N}}$ . Then

$$p_\Omega(x, y; t) = \sum_{k=1}^{\infty} e^{-t\lambda_k} \Phi_k(x) \Phi_k(y), \quad (5.1)$$

in the sense that

$$e^{-tH_\Omega} f = \sum_{k=1}^{\infty} e^{-t\lambda_k} \int_\Omega \Phi_k(y) f(y) \, dy \, \Phi_k$$

for all  $f \in L_2(\Omega)$ , with convergence of the sum in  $L_2(\Omega)$ . Assuming in addition that  $\sum_{k=1}^{\infty} e^{-t\lambda_k} \|\Phi_k\|_1^2 < \infty$ , one obtains that the series (5.1) also converges absolutely in  $L_1(\Omega \times \Omega)$ , and thus

$$Q_\Omega(t) = \sum_{k=1}^{\infty} e^{-t\lambda_k} \left( \int_\Omega \Phi_k(x) \, dx \right)^2 \leq \sum_{k=1}^{\infty} e^{-t\lambda_k} \|\Phi_k\|_1^2. \quad (5.2)$$

The *trace* of the heat semigroup, denoted by  $Z_\Omega(t)$  and defined by

$$Z_\Omega(t) := \sum_{k=1}^{\infty} e^{-t\lambda_k} = \int_\Omega p_\Omega(x, x; t) \, dx,$$

has been studied in great detail too ([14]). It is well-known that heat content or heat trace may be finite for all  $t > 0$  even if the volume of  $\Omega$  is infinite. See for example [5] for an early paper on this subject. The main result of this section reads as follows.

**5.1 Theorem.** *Let  $\Omega$  be an open set in  $\mathbb{R}^d$  such that  $H_\Omega$  has compact resolvent. Then  $Z_\Omega(t) < \infty$  for all  $t > 0$  if and only if  $Q_\Omega(t) < \infty$  for all  $t > 0$ . In either case we have that both*

$$Z_\Omega(t) \leq (2\pi t)^{-d/2} Q_\Omega(t/2), \quad (5.3)$$

and

$$Q_\Omega(t) \leq \hat{C} \left( \lambda_1^{-3d/2} t^{-d} Z_\Omega(t/6)^3 + \lambda_1^{(6-9d)/2} t^{3-4d} Z_\Omega(t/2) \right), \quad (5.4)$$

where  $\hat{C}$  is a constant depending upon  $d$  only.

In the proof of this result it will be shown that the hypothesis that  $Z_\Omega(t) < \infty$  for all  $t > 0$  implies that  $\sum_{k=1}^{\infty} e^{-t\lambda_k} \|\Phi_k\|_1^2 < \infty$  for all  $t > 0$ , and therefore the expression for  $Q_\Omega(t)$  stated in (5.2) is valid.

We will need the following lemma where we use the above notation and the assumptions of Theorem 5.1.

**5.2 Lemma.** *For any  $T > 0$ , we have*

$$N_{2\lambda_k} \leq Z_\Omega(T) e^{2T\lambda_k}. \quad (5.5)$$

*Proof.* From

$$Z_\Omega(T) \geq \sum_{j=1}^k e^{-T\lambda_j} \geq \sum_{j=1}^k e^{-T\lambda_k} = k e^{-T\lambda_k},$$

we get

$$\lambda_k \geq T^{-1} \log \frac{k}{Z_\Omega(T)},$$

and thus

$$N_{2\lambda_k} = |\{j : \lambda_j \leq 2\lambda_k\}| \leq \left| \left\{ j : T^{-1} \log \frac{j}{Z_\Omega(T)} \leq 2\lambda_k \right\} \right| \leq Z_\Omega(T) e^{2T\lambda_k}.$$

which concludes the proof of (5.5). ■

*Proof of Theorem 5.1.* The proof of (5.3) is an immediate consequence of Lemma 2.6 in [5].

The proof of (5.4) relies on the  $L_1$ -bounds for the eigenfunctions in Theorem 0.1 or Corollary 1.4 (with  $\vartheta = 1$ ) which gives the estimate

$$\|\Phi_k\|_1^2 \leq C \lambda_1^{-3d/2} \lambda_k^d \left( (\log N_{2\lambda_k})^d N_{2\lambda_k} + \left( \frac{\lambda_k}{\lambda_1} \right)^{3(d-1)} \right). \quad (5.6)$$

Hence by (5.2) and (5.6) we have that

$$Q_\Omega(t) \leq C \lambda_1^{-3d/2} \sum_{k=1}^{\infty} e^{-t\lambda_k} \lambda_k^d \left( (\log N_{2\lambda_k})^d N_{2\lambda_k} + \left( \frac{\lambda_k}{\lambda_1} \right)^{3(d-1)} \right). \quad (5.7)$$

It is easily seen that  $\log x \leq dx^{1/d}$  ( $x \geq 1$ ) so that

$$Q_\Omega(t) \leq C \lambda_1^{-3d/2} \sum_{k=1}^{\infty} e^{-t\lambda_k} \lambda_k^d \left( d^d N_{2\lambda_k}^2 + \left( \frac{\lambda_k}{\lambda_1} \right)^{3(d-1)} \right). \quad (5.8)$$

The following inequality is useful to bound the polynomial terms in  $\lambda_k$  in (5.8):

$$e^{-tx} x^\alpha \leq (\alpha/e)^\alpha t^{-\alpha} \quad (x > 0, t > 0, \alpha > 0). \quad (5.9)$$

The application of this inequality with  $x = \lambda_k/2$  and  $\alpha = 4d - 3$  gives that

$$\sum_{k=1}^{\infty} e^{-t\lambda_k} \lambda_k^{4d-3} \leq ((8d-6)/e)^{4d-3} t^{3-4d} Z_\Omega(t/2).$$

Hence the second term in (5.8) is bounded by

$$((8d-6)/e)^{4d-3} C \lambda_1^{(6-9d)/2} t^{3-4d} Z_\Omega(t/2). \quad (5.10)$$

By Lemma 5.2, the first term in (5.8) is bounded by

$$\begin{aligned} d^d C \lambda_1^{-3d/2} Z_\Omega(T)^2 \sum_{k=1}^{\infty} e^{-t\lambda_k + 4T\lambda_k} \lambda_k^d \\ \leq d^d (6d/e)^d C \lambda_1^{-3d/2} Z_\Omega(T)^2 t^{-d} \sum_{k=1}^{\infty} e^{-5t\lambda_k/6 + 4T\lambda_k}, \end{aligned} \quad (5.11)$$

where we have used (5.9) with  $x = \lambda_k/6$  and  $\alpha = d$ . We next choose  $T = t/6$  so that the right hand side in (5.11) equals

$$(6d^2/e)^d C \lambda_1^{-3d/2} Z_\Omega(t/6)^3 t^{-d}. \quad (5.12)$$

Putting the two contributions under (5.10) and (5.12) together one obtains the bound under (5.4) with

$$\hat{C} = C \max\{(6d^2/e)^d, ((8d-6)/e)^{4d-3}\}. \quad \blacksquare$$

We finally give a proof of Theorem 1.6.

*Proof of Theorem 1.6.* Using the domain monotonicity of the Dirichlet heat kernel  $0 \leq p_\Omega(x, y; t) \leq p_{\mathbb{R}^d}(x, y; t)$  and the Schwarz inequality, we first obtain

$$\begin{aligned} e^{-t\lambda} \Phi(x) &= e^{-tH_\Omega} \Phi(x) = \int_{\Omega} p_\Omega(x, y; t) \Phi(y) dy \\ &\leq \int_{\Omega} p_\Omega(x, y; t) |\Phi(y)| dy \leq \left( \int_{\Omega} p_{\mathbb{R}^d}^2(x, y; t) dy \right)^{1/2} \|\Phi\|_2, \end{aligned}$$

where  $\int_{\Omega} p_{\mathbb{R}^d}^2(x, y; t) dy = (8\pi t)^{-d/2}$  since  $(2\pi t)^{-d/2} \int_{\mathbb{R}^d} e^{-|x-y|^2/(2t)} dy = 1$ . The choice of  $t$  as  $t := \frac{d}{4\lambda}$  then leads to the desired estimate. Furthermore,  $\|\Phi\|_2^2 = \int_{\Omega} |\Phi|^2 dx \leq \|\Phi\|_\infty \|\Phi\|_1$ , so that (1.8) implies (1.9).  $\blacksquare$

## References

- [1] W. Arendt, Gaussian estimates and interpolation of the spectrum in  $L^p$ . *Differential Integral Equations* **7** (1994), 1153–1168.
- [2] J. Auterhoff, Interpolationseigenschaften des Spektrums linearer Operatoren auf  $L^p$ -Räumen. *Math. Zeitschr.* **184** (1983), 397–406.
- [3] M. van den Berg, On the spectral counting function for the Dirichlet Laplacian. *J. Funct. Anal.* **107** (1992), 352–361.



- [4] M. van den Berg,  $L^p$  bounds for Dirichlet eigenfunctions (Open Problem Session, Problem (v)). In *Geometrical Aspects of Spectral Theory*, Oberwolfach Reports **9**, 2013–2076. European Mathematical Society (EMS), Zürich, 2012.
- [5] M. van den Berg and E. B. Davies, Heat Flow out of Regions in  $\mathbb{R}^m$ . *Math. Zeitschr.* **202** (1989), 463–482.
- [6] M. van den Berg and P. B. Gilkey, Heat content asymptotics of a Riemannian manifold with boundary. *J. Funct. Anal.* **120** (1994), 48–71.
- [7] H. L. Cycon, R. G. Froese, W. Kirsch and B. Simon, *Schrödinger operators*. Springer-Verlag, Berlin, 1987.
- [8] E. B. Davies, *Heat kernels and spectral theory*. Cambridge University Press, Cambridge, 1989.
- [9] E. B. Davies, B. Simon and M. L. Taylor,  $L^p$  spectral theory of Kleinian groups. *J. Funct. Anal.* **78** (1988), 116–136.
- [10] P. A. Deift and R. Hempel, On the existence of eigenvalues of the Schrödinger operator  $H - \lambda W$  in a gap of  $\sigma(H)$ . *Comm. Math. Phys.* **103** (1986), 461–490.
- [11] P. Deift, W. Hunziker, B. Simon, and E. Vock, Pointwise bounds on eigenfunctions and wave packets in  $N$ -body quantum systems IV. *Comm. Math. Phys.* **64** (1978), 1–34.
- [12] M. S. P. Eastham and H. Kalf, *Schrödinger-type operators with continuous spectra*. Research Notes in Mathematics, **65**. Pitman, Boston, Mass.-London, 1982.
- [13] C. Fefferman, The uncertainty principle. *Bull. Amer. Math. Soc. (N.S.)* **9** (1983), 129–206.
- [14] P. B. Gilkey, *Asymptotic Formulae in Spectral Geometry*. Chapman & Hall / CRC, Boca Raton, 2004.
- [15] A. Grigor’yan, *Heat kernel and analysis on manifolds*. AMS/IP Studies in Advanced Mathematics, **47**. American Mathematical Society, Providence, RI; International Press, Boston, MA, 2009.
- [16] R. Hempel and J. Voigt, The spectrum of a Schrödinger operator in  $L_p(\mathbb{R}^\nu)$  is  $p$ -independent. *Comm. Math. Phys.* **104** (1986), 243–250.
- [17] R. Hempel and J. Voigt, On the  $L_p$ -spectrum of Schrödinger operators. *J. Math. Anal. Appl.* **121** (1987), 138–159.
- [18] P. D. Hislop and I. M. Sigal, *Introduction to spectral theory. With applications to Schrödinger operators*. Applied Mathematical Sciences, **113**. Springer-Verlag, New York, 1996.
- [19] T. Kato, *Perturbation theory for linear operators*. Springer-Verlag, Berlin, 1966.

- [20] V. G. Mazy'a, *Sobolev spaces with applications to elliptic partial differential equations*. Grundlehren der Mathematischen Wissenschaften **342**, Springer, Heidelberg, 2011.
- [21] V. G. Mazy'a, On the theory of the higher-dimensional Schrödinger operator. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* **28** (1964), 1145–1172.
- [22] S. Port and C. Stone, *Brownian motion and classical potential theory*. Probability and Mathematical Statistics. Academic Press, New York-London, 1978.
- [23] O. Post, Eigenvalues in spectral gaps of a perturbed periodic manifold. *Mathematische Nachrichten* **261–262** (2003), 141–162.
- [24] P. Stollmann and J. Voigt, Perturbation of Dirichlet forms by measures. *Potential Analysis* **5** (1996), 109–138.
- [25] K. T. Sturm, On the  $L^p$ -spectrum of uniformly elliptic operators on Riemannian manifolds. *J. Funct. Anal.* **118** (1993), 442–453.
- [26] H. Vogt,  $L_1$ -estimates for eigenfunctions and heat kernel estimates for semigroups dominated by the free heat semigroup. Preprint 2014.
- [27] K. Yoshitomi, Band gap of the spectrum in periodically curved quantum waveguides. *J. Differential Equations* **142** (1998), 123–166.